

Rolling friction for hard cylinder and sphere on viscoelastic solid

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Abstract. We calculate the friction force acting on a hard cylinder or spherical ball rolling on a flat surface of a viscoelastic solid. The rolling-friction coefficient depends non-linearly on the normal load and the rolling velocity. For a cylinder rolling on a viscoelastic solid characterized by a single relaxation time Hunter has obtained an exact result for the rolling friction, and our result is in very good agreement with his result for this limiting case. The theoretical results are also in good agreement with experiments of Greenwood and Tabor. We suggest that measurements of rolling friction over a wide range of rolling velocities and temperatures may constitute a useful way to determine the viscoelastic modulus of rubber-like materials.

1 Introduction

Rubber friction is a topic of huge practical importance, *e.g.*, for tires, rubber seals, wiper blades, conveyor belts and syringes [1–16]. Many experiments have been performed with a hard spherical ball rolling on a flat rubber substrate [17–20]. Nearly the same friction force is observed during sliding as during rolling, assuming that the interface is lubricated and that the sliding velocity and fluid viscosity are such that a thin lubrication film is formed with a thickness much smaller than the indentation depth of the ball, but larger than the amplitude of the roughness on the surfaces [17]. The results of rolling friction experiments have often been analyzed using a very simple model of Greenwood and Tabor [17], which however contains an (unknown) factor α , which represents the fraction of the input elastic energy lost as a result of the internal friction of the rubber. In this paper we present a very simple theory for the friction force acting on a hard cylinder or spherical ball rolling on a flat rubber surface. For a cylinder rolling on a viscoelastic solid characterized by a single relaxation time Hunter [21] has obtained an exact result for the rolling friction, and our result is in very good agreement with his result for this limiting case.

2 Theory

Using the theory of elasticity (assuming an isotropic viscoelastic medium), one can calculate the displacement field u_i on the surface $z = 0$ in response to the surface

stress distributions $\sigma_i = \sigma_{3i}$. Let us define the Fourier transform

$$u_i(\mathbf{q}, \omega) = \frac{1}{(2\pi)^3} \int d^2x dt u_i(\mathbf{x}, t) e^{-i(\mathbf{q}\cdot\mathbf{x} - \omega t)}$$

and similar for $\sigma_i(\mathbf{q}, \omega)$. Here $\mathbf{x} = (x, y)$ and $\mathbf{q} = (q_x, q_y)$ are two-dimensional vectors. In ref. [3] we have shown that

$$u_i(\mathbf{q}, \omega) = M_{ij}(\mathbf{q}, \omega) \sigma_j(\mathbf{q}, \omega),$$

or, in matrix form,

$$\mathbf{u}(\mathbf{q}, \omega) = M(\mathbf{q}, \omega) \boldsymbol{\sigma}(\mathbf{q}, \omega),$$

where the matrix M is given in ref. [3].

We now assume that $|\nabla u_z(\mathbf{x})| \ll 1$ and that the surface stress only acts in the z -direction, so that

$$u_z(\mathbf{q}, \omega) = M_{zz}(\mathbf{q}, \omega) \sigma_z(\mathbf{q}, \omega). \quad (1)$$

Since in the present case ω is of order vq (where v is the sliding or rolling velocity) we get $\omega/c_T q = v/c_T \ll 1$ in most cases of practical interest, where c_T is the transverse sound velocity in the rubber. In this case (see ref. [3])

$$(M_{zz})^{-1} = -\frac{Eq}{2(1-\nu^2)}. \quad (2)$$

It is interesting to note that if, instead of assuming that the surface stress acts in the z -direction, we assume that the displacement \mathbf{u} points along the z -direction, then

$$\sigma_z(\mathbf{q}, \omega) = (M^{-1})_{zz}(\mathbf{q}, \omega) u_z(\mathbf{q}, \omega),$$

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where in the limit $\omega/c_T q \ll 1$,

$$(M^{-1})_{zz} = -\frac{2Eq(1-\nu)}{(1+\nu)(3-4\nu)},$$

which differs from (2) only with respect to a factor $4(1-\nu)^2/(3-4\nu)$. For rubber-like materials ($\nu \approx 0.5$) this factor is of order unity. Hence, practically identical results are obtained independently of whether one assumes that the interfacial stress or displacement vector is perpendicular to the nominal contact surface. In reality, neither of these two assumptions holds strictly, but the result above indicates that the theory is not sensitive to this approximation.

Now, assume that

$$\sigma_z(\mathbf{x}, t) = \sigma_z(\mathbf{x} - \mathbf{v}t),$$

then

$$\begin{aligned} \sigma_z(\mathbf{q}, \omega) &= \frac{1}{(2\pi)^3} \int d^2x dt \sigma_z(\mathbf{x} - \mathbf{v}t) e^{-i(\mathbf{q}\cdot\mathbf{x} - \omega t)} \\ &= \delta(\omega - \mathbf{q} \cdot \mathbf{v}) \sigma_z(\mathbf{q}), \end{aligned} \quad (3)$$

where

$$\sigma_z(\mathbf{q}) = \frac{1}{(2\pi)^2} \int d^2x \sigma_z(\mathbf{x}) e^{-i\mathbf{q}\cdot\mathbf{x}}.$$

If F_f denotes the friction force, then the energy dissipated during the time period t_0 equals

$$\Delta E = F_f v t_0. \quad (4)$$

But this energy can also be written as

$$\begin{aligned} \Delta E &= \int d^2x dt \dot{\mathbf{u}} \cdot \boldsymbol{\sigma} \\ &= (2\pi)^3 \int d^2q d\omega (-i\omega) \mathbf{u}(\mathbf{q}, \omega) \cdot \boldsymbol{\sigma}(-\mathbf{q}, -\omega), \end{aligned} \quad (5)$$

where $\omega = \mathbf{v} \cdot \mathbf{q}$. Substituting (1) in (5) and using (3) and that (for large t_0)

$$\begin{aligned} [\delta(\omega - \mathbf{q} \cdot \mathbf{v})]^2 &= \delta(\omega - \mathbf{q} \cdot \mathbf{v}) \frac{1}{2\pi} \int dt e^{i(\omega - \mathbf{q} \cdot \mathbf{v})t} \\ &= \delta(\omega - \mathbf{q} \cdot \mathbf{v}) \frac{1}{2\pi} \int_{-t_0/2}^{t_0/2} dt \\ &= \delta(\omega - \mathbf{q} \cdot \mathbf{v}) (t_0/2\pi), \end{aligned}$$

gives

$$\begin{aligned} \Delta E &= (2\pi)^2 t_0 \int d^2q (-i\omega) \\ &\quad \times M_{zz}(-\mathbf{q}, -\omega) \sigma_z(\mathbf{q}) \sigma_z(-\mathbf{q}). \end{aligned}$$

Comparing this expression with (4) gives the friction force

$$F_f = \frac{(2\pi)^2}{v} \int d^2q (-i\omega) M_{zz}(-\mathbf{q}, -\omega) \sigma_z(\mathbf{q}) \sigma_z(-\mathbf{q}).$$

Since F_f is real and since $\sigma_z(-\mathbf{q}) = \sigma_z^*(\mathbf{q})$, we get

$$F_f = \frac{(2\pi)^2}{v} \int d^2q \omega \operatorname{Im} M_{zz}(-\mathbf{q}, -\omega) |\sigma_z(\mathbf{q})|^2. \quad (6a)$$

Using (2) we can also write

$$F_f = \frac{2(2\pi)^2}{v} \int d^2q \frac{\omega}{q} \operatorname{Im} \frac{1}{E_{\text{eff}}(\omega)} |\sigma_z(\mathbf{q})|^2, \quad (6b)$$

where $E_{\text{eff}} = E/(1-\nu^2)$. In principle, ν depends on the frequency but the factor $1/(1-\nu^2)$ varies from $4/3 \approx 1.33$ for $\nu = 0.5$ (rubbery region) to ≈ 1.19 for $\nu = 0.4$ (glassy region) and we can neglect the weak dependence on frequency. Within the assumptions given above, eq. (6) is exact. Note that even if we use $\sigma_z(\mathbf{q})$ calculated to zero order in $\tan \delta$, the friction force (6) will be correct to linear order in $\tan \delta$. We also note that the present approach is very general and flexible. For example, instead of a semi-infinite solid as assumed above one may be interested in a thin viscoelastic film on a hard flat substrate. In this case, (assuming slip-boundary conditions) the $M(\mathbf{q}, \omega)$ -function which enters in (6a) is determined by [8, 22]

$$\begin{aligned} M^{-1} &= -\frac{Eq}{2(1-\nu^2)} S, \\ S &= \frac{(3-4\nu) \cosh(2qd) + 2(qd)^2 - 4\nu(3-2\nu) + 5}{(3-4\nu) \sinh(2qd) - 2qd}. \end{aligned}$$

Note that as $d \rightarrow \infty$, $S \rightarrow 1$ and the present result reduces to (2). Substituting this in (6a) gives the rolling friction for a sphere (or cylinder) on a thin rubber film adsorbed to a hard flat substrate. We now apply the theory to a) a rigid cylinder and b) a rigid sphere rolling on a semi-infinite viscoelastic solid.

2.1 Cylinder

Consider a hard cylinder (radius R and length $L_y \gg R$) rolling on a viscoelastic solid. The same result is obtained during sliding if one assumes lubricated contact and if one can neglect the viscous energy dissipation in the lubrication film. As discussed above, when calculating the friction force to linear order in $\tan \delta$ we can neglect dissipation when calculating the contact pressure $\sigma_z(x)$ and assume that the stress is of the Hertz form. Thus if we introduce a coordinate system with the y -axis parallel to the cylinder axis and with the origin of the x -axis in the middle of the contact area (of width $2a$), then the contact stress for $-a < x < a$ is

$$\sigma_z(x) = \frac{2f_N}{\pi a} \left[1 - \left(\frac{x}{a} \right)^2 \right]^{1/2}, \quad (7)$$

where $f_N = F_N/L_y$ the load per unit length of the cylinder. The half-width of the contact area in the Hertz contact theory is

$$a = \left(\frac{4f_N R}{\pi E_{\text{eff}}} \right)^{1/2}, \quad (8)$$

where we take E_{eff} to be $|E_{\text{eff}}(\omega)|$, with $\omega = \mathbf{q} \cdot \mathbf{v}$. Now, note that

$$\begin{aligned} & \int d^2x \left[1 - \left(\frac{x}{a} \right)^2 \right]^{1/2} e^{-i\mathbf{q} \cdot \mathbf{x}} \\ &= 2\pi\delta(q_y) \int_{-a}^a dx \left[1 - \left(\frac{x}{a} \right)^2 \right]^{1/2} e^{-iq_x x} \\ &= 2\pi\delta(q_y)a \int_{-1}^1 dz \left[1 - z^2 \right]^{1/2} e^{-iq_x a z} \\ &= 2\pi\delta(q_y) \frac{\pi}{q_x} J_1(aq_x). \end{aligned} \quad (9)$$

Using (7) and (9) gives

$$\sigma(\mathbf{q}) = \frac{f_N}{\pi a q_x} \delta(q_y) J_1(q_x a). \quad (10)$$

Note that this expression satisfies

$$(2\pi)^2 \sigma(\mathbf{q} = \mathbf{0}) = \int d^2x \sigma_z(\mathbf{x}) = F_N,$$

where we have used that $\delta(q_y = 0) = L_y/2\pi$. Substituting (10) in (6b) dividing by F_N , and using that

$$[\delta(q_y)]^2 = (L_y/2\pi)\delta(q_y),$$

gives the friction coefficient

$$\mu = \frac{8f_N}{\pi} \int_0^\infty dq_x \text{Im} \frac{1}{E_{\text{eff}}(q_x v)} \frac{1}{(aq_x)^2} J_1^2(q_x a). \quad (11)$$

Let us now assume the simplest possible viscoelastic modulus characterized by a single relaxation time τ

$$E_{\text{eff}}(\omega) = \frac{E_1(1 - i\omega\tau)}{E_1/E_0 - i\omega\tau}, \quad (12)$$

where E_1/E_0 is the ratio between the high-frequency and the low-frequency modulus. In fig. 1 we show the friction coefficient (times the radius R of the cylinder and divided by the half-width $a_0 = a(v = 0)$ of the static contact region) as a function of $v\tau/a_0$, where v is the rolling velocity. We have assumed $E_1/E_0 = 10$. We compare the exact result (blue curve) of Hunter [21] (the same result was obtained by Goryacheva [23]) with the prediction of eq. (11) (green curve). Note that some distance away from the maximum the agreement between the two curves is perfect. This is expected because these regions correspond to small $\tan \delta$ where eq. (11) should be essentially exact. Close to the maximum a small difference occurs between the two curves, but from a practical point of view this is not important, since real rubber exhibits some

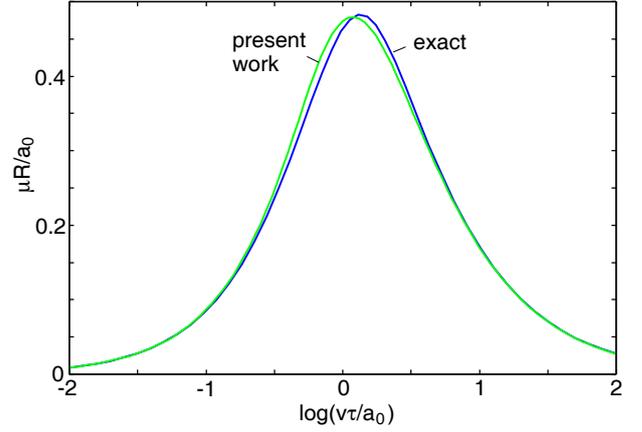


Fig. 1. (Colour on-line) The friction coefficient (times the radius R of the cylinder and divided by the half-width a_0 of the static contact region) as a function of $v\tau/a_0$, where v is the rolling velocity, τ the rubber viscoelastic relaxation time (for $E_1/E_0 = 10$). We compare the exact result (blue curve) of Hunter [21] with the prediction of eq. (11) (green curve).

non-linearity making any linear viscoelasticity theory only approximately valid anyhow.

At this point we empathize that (6) is basically exact, and if one could calculate the contact pressure $\sigma_z(x)$ (or rather the Fourier transform $\sigma_z(q_x)$) exactly, then (6) should give the exact result, *e.g.*, the result of Hunter for the cylinder case, and assuming the simple viscoelastic modulus (12). Note that the contact pressure will not be symmetric around the midpoint when the dissipation in the rubber is included in the analysis. Still, to linear order in $\tan \delta$ one can neglect this effect (since (6) is explicitly already linear in $\tan \delta$), and the analysis above shows that this is a remarkable accurate approximation, which is an interesting result in its own right, and makes it possible to apply the theory to a wide set of problems.

2.2 Sphere

Consider a hard spherical (radius R) ball rolling on a viscoelastic solid. As discussed above, when calculating the friction force to linear order in $\tan \delta$ we can neglect dissipation when calculating the contact pressure $\sigma_z(\mathbf{x})$ and assume that the stress is of the Hertz form

$$\sigma_z(\mathbf{x}) = \frac{3F_N}{2\pi r_c^2} \left[1 - \left(\frac{r}{r_c} \right)^2 \right]^{1/2}, \quad (13)$$

where $r = |\mathbf{x}|$ is the distance from the center of the contact area and F_N the load. The radius of the contact area in the Hertz contact theory is

$$r_c = \left(\frac{3F_N R}{4E_{\text{eff}}} \right)^{1/3}, \quad (14)$$

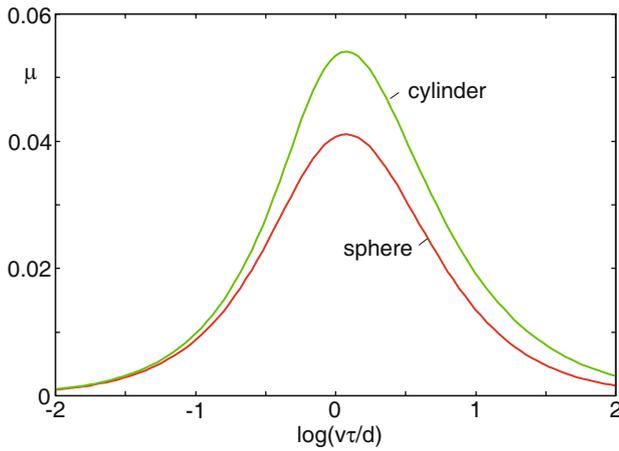


Fig. 2. The friction coefficient as a function of $v\tau/d$, where v is the rolling velocity, τ the rubber viscoelastic relaxation time, and $d = a_0$ for the cylinder (upper curve) or $d = r_0$ for the sphere (lower curve), rolling on a rubber substrate described by a simple viscoelastic model (see eq. (12)). Here a_0 is half the width of the cylinder-substrate contact area and r_0 the radius of the contact region for the sphere, both in the limit of vanishing rolling velocity. In the calculation the radius of the sphere and of the cylinder are both $R = 1$ cm and the load was chosen so that the average contact pressure for the stationary contact was the same (equal to about 44 kPa). For $E_0 = 1$ MPa and $E_1/E_0 = 10$.

where we take E_{eff} to be $|E_{\text{eff}}(\omega)|$, with $\omega = \mathbf{q} \cdot \mathbf{v}$. Now, note that

$$\begin{aligned}
 & \int d^2x \left[1 - \left(\frac{r}{r_c} \right)^2 \right]^{1/2} e^{-i\mathbf{q} \cdot \mathbf{x}} \\
 &= \int_0^{r_c} dr r \left[1 - \left(\frac{r}{r_c} \right)^2 \right]^{1/2} \int_0^{2\pi} d\phi e^{-iqr \cos \phi} \\
 &= 2\pi \int_0^{r_c} dr r \left[1 - \left(\frac{r}{r_c} \right)^2 \right]^{1/2} J_0(qr) \\
 &= 2\pi r_c^2 \int_0^1 dz z [1 - z^2]^{1/2} J_0(qr_c z) \\
 &= 2\pi r_c^2 (qr_c)^{-3} [\sin(qr_c) - qr_c \cos(qr_c)]. \quad (15)
 \end{aligned}$$

Using (13) and (15) gives

$$\sigma(\mathbf{q}) = \frac{3F_N}{4\pi^2} \frac{1}{(qr_c)^3} [\sin(qr_c) - qr_c \cos(qr_c)]. \quad (16)$$

Note that this expression satisfies

$$(2\pi)^2 \sigma(\mathbf{q} = \mathbf{0}) = \int d^2x \sigma_z(\mathbf{x}) = F_N.$$

Substituting (16) in (6b) and dividing by F_N gives the friction coefficient

$$\begin{aligned}
 \mu &= \frac{9F_N}{2\pi^2} \int_0^\infty dq q \int_0^{2\pi} d\phi \cos \phi \operatorname{Im} \frac{1}{E_{\text{eff}}(qv \cos \phi)} \\
 &\quad \times \frac{1}{(qr_c)^6} [\sin(qr_c) - qr_c \cos(qr_c)]^2. \quad (17)
 \end{aligned}$$

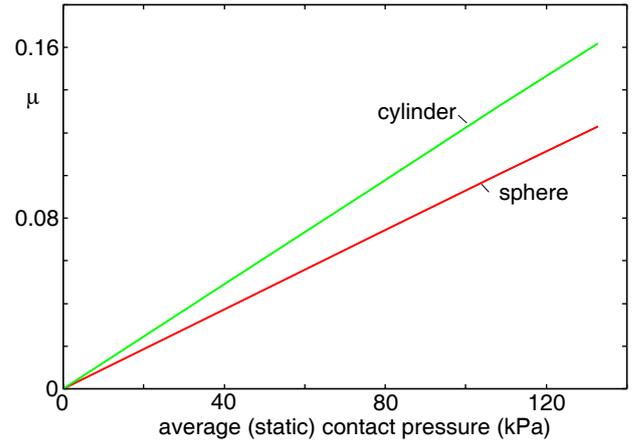


Fig. 3. The maximum (as a function of the velocity v) of the rolling-friction coefficients for a sphere and a cylinder, as a function of the average (static) contact pressure. For the same parameters as in fig. 2.

Equation (17) is very general with no restriction on the viscoelastic properties of the rubber or on the sliding velocity v (assuming that v is small enough that the effect of frictional heating is negligible, and assuming $v \ll c_T$).

In fig. 2 we show the rolling-friction coefficient as a function of $v\tau/r_0$ (where r_0 is the static contact radius) for a sphere, and as a function of $v\tau/a_0$ for a cylinder. The ball has radius $R = 1$ cm and is squeezed against a rubber surface with load $F_N = 0.15$ N, giving a static contact area with radius $r_0 = r_c(v = 0) = 0.10$ cm. The average contact pressure $p_a = F_N/(\pi r_0^2) = 44.3$ kPa is the same as for the cylinder case shown in the same figure. The cylinder has radius $R = 1$ cm and is squeezed against a rubber surface with load $f_N = 100$ N/m, giving a static contact area with width $a_0 = 0.11$ cm. The average contact pressure is $p_a = f_N/(2a_0) = 44.3$ kPa. In the calculation we have used the viscoelastic modulus $E_{\text{eff}}(\omega)$ given in eq. (12). Note that at very low sliding velocities where the (average) contact pressures are the same the rolling frictions are nearly identical.

In fig. 3 we show the maximum (as a function of the velocity v) of the rolling-friction coefficients for a sphere and a cylinder, as a function of the average pressure in the static contact area. Note that μ varies nearly linearly with the (static) contact pressure, and using (8) and (14) it follows that the rolling-friction coefficient $\mu \sim f_N^{1/2}$ for the cylinder and $\mu \sim F_N^{1/3}$ for the sphere.

Let us now consider the limiting case when the rolling velocity v is so small that only the low-frequency viscoelastic modulus is relevant. We also assume that $E(\omega)$ is given by (12). If $\omega\tau \ll 1$ for all relevant frequencies, we get $\operatorname{Re} E(\omega) \approx E(0) = E_0$ and $\operatorname{Im} E(\omega) \approx -\omega\tau E_0(1 - E_0/E_1)$. We also get $|E(\omega)| \approx E_0$. Substituting these results in (17) gives after some simplifications

$$\mu \approx \frac{9I}{2} \frac{p_a}{E_0} \left(1 - \frac{E_0}{E_1} \right) \frac{v\tau}{r_0}, \quad (18)$$

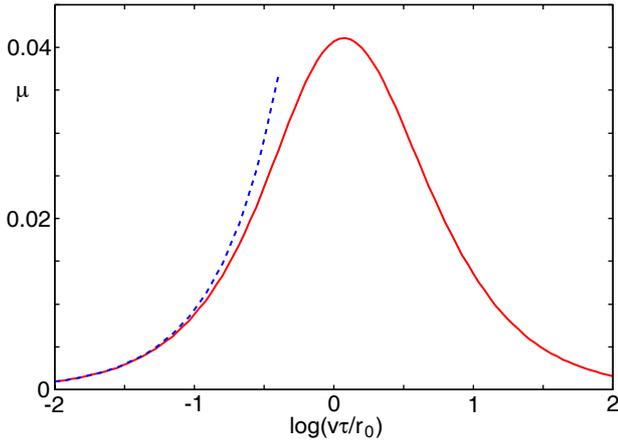


Fig. 4. The friction coefficient as a function of $v\tau/r_0$ for the sphere. The full line is using the full theory (eq. (17)) and the dashed line using the asymptotic, low velocity, result (19). For the same parameters as in fig. 2.

where $p_a = F_N/(\pi r_0^2)$ is the (average) static contact pressure, and where

$$I = \int_0^\infty d\xi \xi^{-4} (\sin \xi - \xi \cos \xi)^2 \approx 0.52.$$

Note that $\mu \sim p_a$ which is consistent with fig. 3. If we define the frequency $\omega = v/r_0$ and note that in the present case $\tan \delta = (1 - E_0/E_1)\omega\tau$, we can write

$$\mu \approx \frac{9I}{2} \frac{p_a}{E_0} \tan \delta.$$

Since $9I/2 \approx 2.34$ we get

$$\mu \approx 2.34 \frac{p_a}{E_0} \tan \delta. \quad (19)$$

In fig. 4 we compare this asymptotic (low velocity) result with the full theory (eq. (17)). For the limiting case studied above, Greenwood and Tabor have shown that

$$\mu = \frac{9\pi}{64} \frac{p_a}{E_0} \alpha, \quad (20)$$

where α is the fraction of the input elastic energy lost as a result of the internal damping of the rubber. Thus we can write

$$\alpha \approx 5.3 \tan \delta. \quad (21)$$

Greenwood and Tabor analyzed rolling (and lubricated sliding) friction data using (20). They obtained the best fit to the experimental data by using an α which was almost a factor of two larger than obtained from cyclic (low frequency) simple uniaxial loading-unloading measurements of rubber strips, where the energy loss due to hysteresis was about 0.35 of the maximal elastic energy of deformation. For a linear viscoelastic solid the latter is given by $\pi \tan \delta$ (see appendix A). Our theory predicts that α (see (21)) is about 1.7 times larger than $\pi \tan \delta$, and using

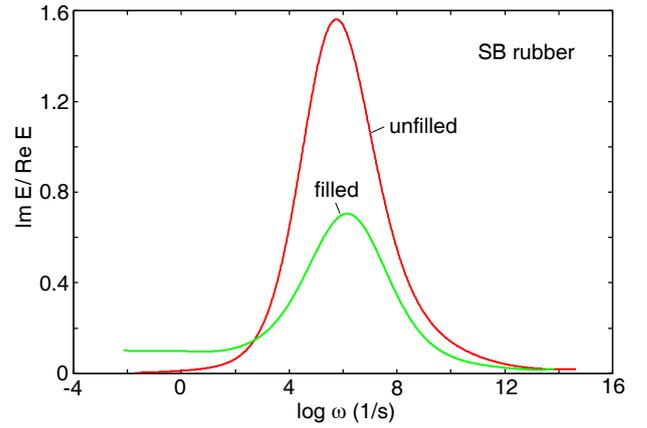


Fig. 5. The loss tangent $\tan \delta = \text{Im } E(\omega)/\text{Re } E(\omega)$ at $T = 20^\circ\text{C}$, as a function of frequency ω for an unfilled and filled SB rubber.

this result gives very good agreement with the measured data presented in ref. [17]. The reason for why rolling-friction experiments result in an α -parameter which is larger than expected from uniaxial tension tests is related to the very different nature of the time-dependent deformations: a rubber volume element below a rolling sphere (or cylinder) undergoes, for some fraction of the interaction time, time-dependent deformations where the total elastic energy is nearly constant while the stress directions are changing and energy is being lost (see discussion in refs. [17,18]).

3 Rolling friction on rubber

We have calculated the rolling friction when a hard cylinder with radius $R = 1$ cm is rolling on a Styrene-Butadiene (SB) copolymer rubber surface at room temperature. The viscoelastic modulus of the rubber has been measured, and in fig. 5 we show the loss tangent $\tan \delta = \text{Im } E(\omega)/\text{Re } E(\omega)$ at $T = 20^\circ\text{C}$, as a function of frequency ω for an unfilled and filled SB rubber. Note that $\tan \delta$ for the filled SB rubber has a tail extending to very low frequencies. This corresponds to relaxation processes in the rubber with very long relaxation times (see fig. B.2 in [4]). This effect results from the increase in the activation barrier for polymer rearrangement processes for rubber molecules bound to (or close to) filler particles (in this case carbon particles).

The rolling-friction coefficient is shown in fig. 6 when the normal force per unit length is 10 N/cm. The rolling friction exhibits a very strong temperature dependence given by the WLF shift factor a_T : reducing the temperature by 10°C results in a shift of the rolling-friction curve towards lower velocities by approximately one decade. At low frequencies (or low rolling velocities) the unfilled SB rubber is elastically much softer than the filled rubber. Thus, the half-width a_0 of the static Hertz contact region is 1.4 mm for the unfilled rubber but only 0.6 mm for the filled rubber. At high rolling velocities these values

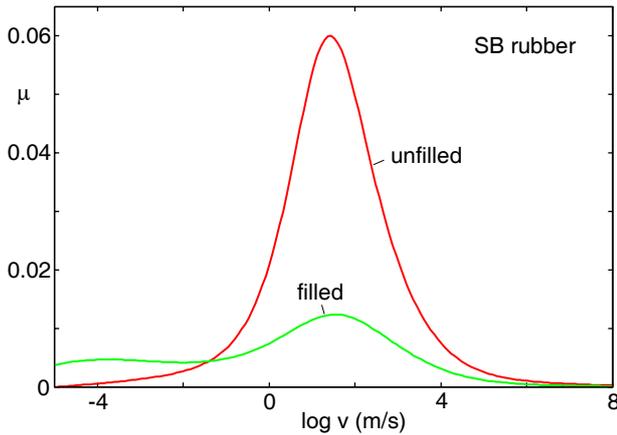


Fig. 6. The rolling-friction coefficient for a hard cylinder with radius $R = 1$ cm rolling on unfilled and filled SB rubber at $T = 20^\circ\text{C}$. The normal force on the cylinder per unit length is 10 N/cm .

becomes much smaller, owing to the stiffening of the effective elastic modulus at high frequencies. Note also that the rolling friction for filled SB rubber has a tail extending towards lower rolling velocities, which has the same origin as the tail in $\tan\delta$ towards lower frequencies, involving rubber molecules bound to the filler particles.

4 Discussion

Consider a rolling cylinder (or ball) in a reference frame where the center-of-mass velocity v vanishes. The rolling friction gives a moment $F_f R$ at the center of the cylinder. Since the shear stress at the interface is assumed to vanish, this moment must arise from a contact pressure $\sigma_z(x)$ which is asymmetric around the center line of the cylinder. That is,

$$F_f R = L_y \int dx x \sigma_z(x).$$

Thus the calculation of the rolling resistance gives information about how the centroid of the contact pressure shifts away from the symmetry line $x = 0$ due to the delayed response of the rubber caused by the internal friction of the rubber (characterized by the relaxation time τ in (12)). This information could in principle be used to improve the theoretical treatment given above, but since the simple theory we use seems to be very accurate, we will not consider this point further here.

The study presented above assumes that the adhesive interaction between the rubber and the hard ball can be neglected. Most rubbers of engineering interest are strongly cross-linked and have fillers making them relative stiff and reducing the role of adhesion. Adhesion can also be removed (or reduced) by lubricating the interface or by adsorbing small inert solid particles on the rubber surface, *e.g.*, talc. In these latter cases, the rolling and sliding friction may be nearly the same as indeed observed in some

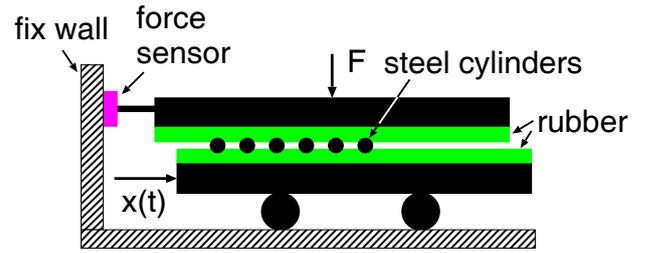


Fig. 7. Proposed experimental set-up for measuring rolling friction. Rubber sheets with a thickness of order 0.5 cm are attached (glued) to both steel plates. Several (say n) steel cylinders with a diameter of order 2 mm are placed between the rubber sheets. The steel blocks are squeezed together with the force F_N , and the bottom steel block moves with a velocity v parallel to the upper steel block. The force on the upper steel block divided by the number n of steel cylinders, and by a factor 2 (to take into account the two rubber sheets) determine the rolling friction between one cylinder and the substrate at the load F_N/n .

experiments [17]. For very soft rubber and for clean surfaces, during rolling or sliding an opening (and a closing) crack will propagate at the interface, and associated with this may be very strong energy dissipation, which may dominate the rolling or sliding friction (see, *e.g.*, ref. [24]). We note finally that measurements of the rolling friction (see fig. 7) as a function of velocity v and temperature T may be a very useful way of determining the viscoelastic modulus $E(\omega)$, see appendix B.

5 Summary and conclusion

We have presented a very general and flexible approach to calculate the rolling resistance of hard objects on viscoelastic solids. The theory can be applied to materials with arbitrary (*e.g.*, measured) viscoelastic modulus $E(\omega)$. The theory can be applied to both spheres and cylinders rolling on semi-infinite viscoelastic solids, or on a thin viscoelastic film adsorbed on a rigid flat substrate, or even more complex situations for which the $M(\mathbf{q}, \omega)$ -function can be calculated. For a cylinder rolling on a viscoelastic solid characterized by a single relaxation time τ , Hunter has obtained an exact solution for the rolling resistance. We have shown that for this limiting case our theory gives almost the same result as obtained by Hunter. We have compared the rolling resistance of a sphere with that of a cylinder under similar circumstances. Measurements of the rolling friction as a function of velocity and temperature may be a very useful way of determining the viscoelastic modulus.

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Appendix A.

Consider a strip of rubber in (slow) loading-unloading and assume that the strain $\varepsilon = \varepsilon_0 \sin(\omega t)$, $0 < t < t_0$, with $\omega t_0 = \pi$. The energy dissipation (per unit volume)

$$\begin{aligned} U_{\text{diss}} &= \int_0^{t_0} dt \sigma \dot{\varepsilon} \\ &= \int_0^{t_0} dt \frac{\varepsilon_0}{2i} (E(\omega)e^{i\omega t} - E(-\omega)e^{-i\omega t}) \\ &\quad \times \frac{\varepsilon_0}{2} \omega (e^{i\omega t} + e^{-i\omega t}) = \frac{1}{2} \varepsilon_0^2 \omega t_0 \text{Im} E(\omega). \end{aligned}$$

We are considering very low frequencies so that the maximum elastic energy

$$U_{\text{el}} \approx \frac{1}{2} E(0) \varepsilon_0^2 \approx \frac{1}{2} \text{Re} E(\omega) \varepsilon_0^2.$$

Thus we get

$$\alpha = \frac{U_{\text{diss}}}{U_{\text{el}}} = \pi \frac{\text{Im} E(\omega)}{\text{Re} E(\omega)} \approx \pi \tan \delta. \quad (\text{A.1})$$

Appendix B.

The viscoelastic modulus $E(\omega)$ is a complex quantity and at first one may think that it impossible to determine both $\text{Re} E(\omega)$ and $\text{Im} E(\omega)$ from the knowledge of a single function $\mu(v)$ which depends on $E(\omega) = \text{Re} E(\omega) + i \text{Im} E(\omega)$. However, $E(\omega)$ is (assumed to be) a causal linear response function so that $\text{Im} E(\omega)$ can in fact be obtained from $\text{Re} E(\omega)$ using a Kramer-Kronig relation. Thus we have one known function $\mu(v)$ and only one unknown function, *e.g.*, $\text{Re} E(\omega)$, and $E(\omega)$ can in principle be determined uniquely from $\mu(v)$. The best way of doing this is to note that $E(\omega)$ can be written as

$$E(\omega) = E_1 - \int_0^\infty d\tau \frac{H(\tau)}{1 - i\omega\tau}. \quad (\text{B.1})$$

In numerical calculations one may discretize relation (B.1) and it is usually enough to include $n \approx 15$ relaxation times τ_k , with $\tau_{k+1} \approx 10\tau_k$. Thus we expand $E(\omega)$ in the form

$$E(\omega) = E_1 - \sum_{k=1}^n \frac{H_k}{1 - i\omega\tau_k}, \quad (\text{B.2})$$

where H_k and the relaxation times τ_k are real positive quantities. Next, let us form the quantity

$$V = \sum_i [\mu_{\text{meas}}(v_i) - \mu_{\text{theory}}(v_i; H_1, \dots, H_n)]^2 g_i, \quad (\text{B.3})$$

where g_i are suitable chosen weight coefficients (*e.g.*, $g_i = 1$) and where v_i are the velocities for which the rolling friction has been measured. In (B.3) we have indicated that the theory expression for the rolling-friction coefficient depends on the numbers H_k . The determination of H_k is a problem in multidimensional minimization of V , and can be performed using different methods, *e.g.*, the Monte Carlo method or the Amoeba method, see ref. [25].

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